Modern techniques in quantum algorithms

QTML 2025 Tutorial

Zane Marius Rossi

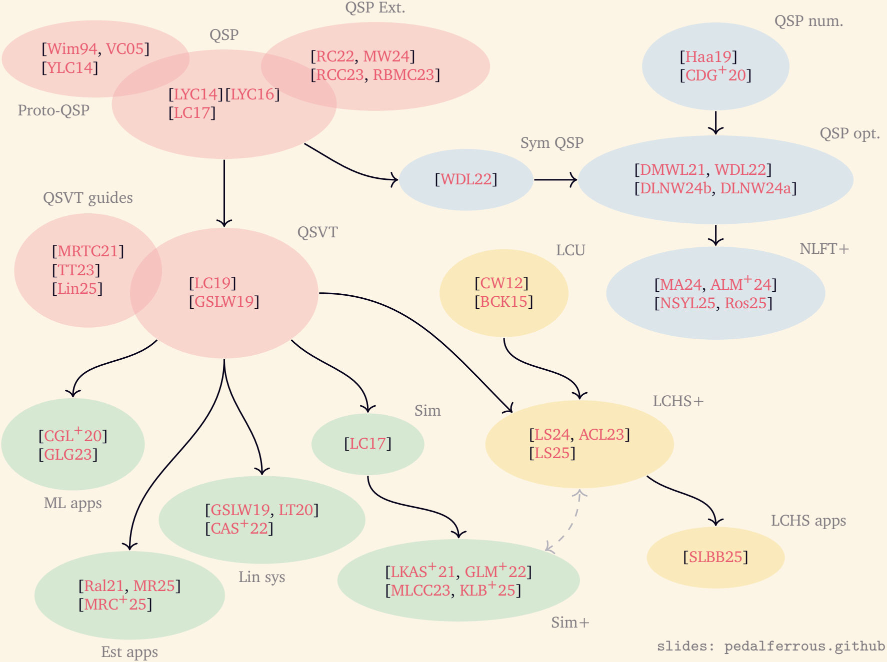
The University of Tokyo

November 16, 2025

🎯 Today's aims

1. Motivation and crash-course for QSP[1] ★
2. Numerical interlude ★
3. Qubitization, block encodings, and QSVT[2] ★
4. Applications and the state of the art ★
5. The limits of QSVT, and alternative approaches ★
6. Open problems, recent progress, and outlook ★★★★

[1] Quantum signal processing
[2] Quantum singular value transformation


QSP Ext.
[RC22, MW24]
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[CGL+20]
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ML apps
Sim
[LC17]
LCHS+
[LS24, ACL23]
[LS25]
[GSLW19, LT20]
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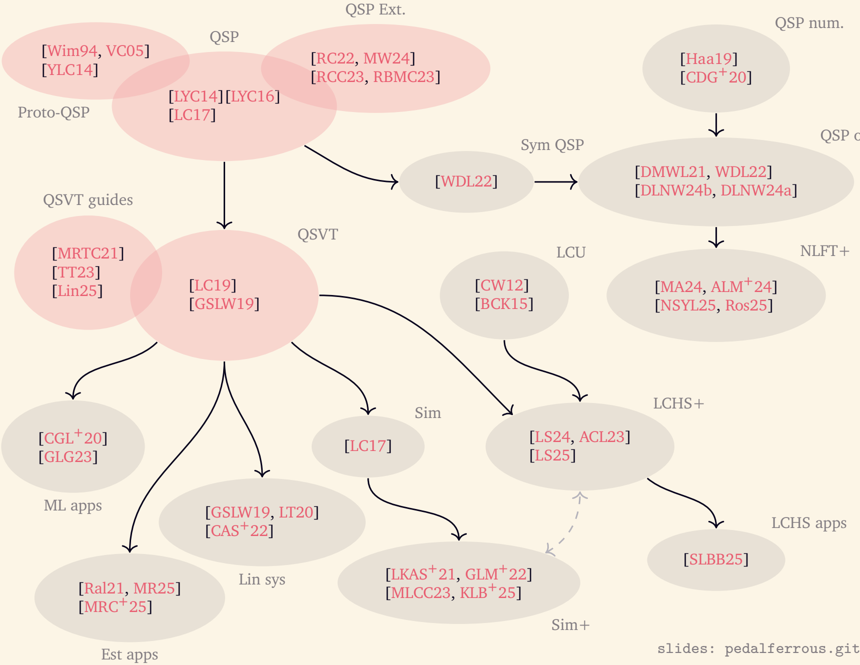

slides: pedalferrous.github.io

⚛️ NMR and composite pulses

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What can we do with many copies of an ***unknown but consistent*** unitary process?[1] [Wim94, VC05, BHC04]

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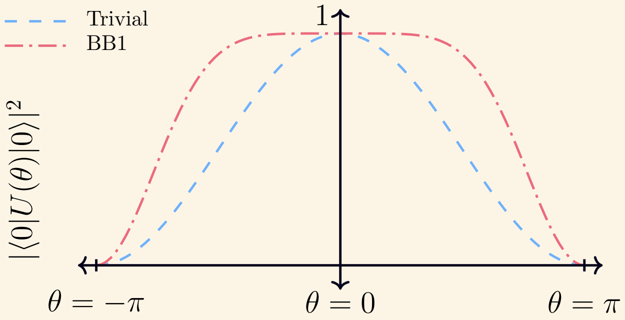

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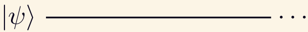
$$W(x) = \begin{bmatrix} x & i\sqrt{1-x^2} \\ i\sqrt{1-x^2} & x \end{bmatrix}, \qquad e^{i\phi} = \begin{bmatrix} e^{i\phi} & \\ & e^{-i\phi} \end{bmatrix}.$$

Properties of the QSP ansatz



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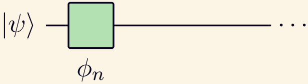

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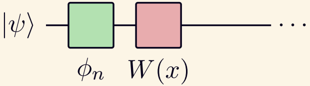

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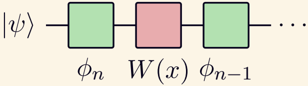

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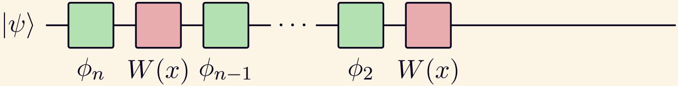

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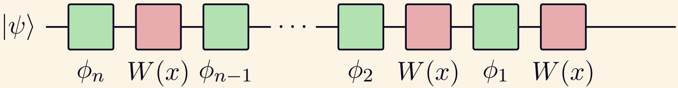

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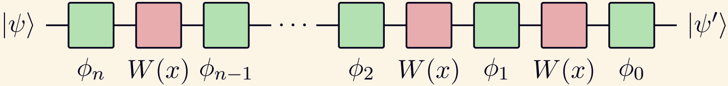

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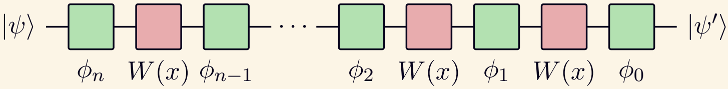

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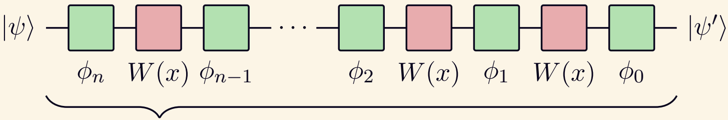

$$U(\Phi, H) = e^{i\phi_0 Z}\prod_{j=1}^{k} W(H)\, e^{i\phi_j Z} = \begin{array}{cc} \begin{bmatrix} P(H) & iQ(H)\sqrt{1-H^2} \\ iQ(H)^*\sqrt{1-H^2} & P(H)^* \end{bmatrix} & \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \\ \begin{array}{cc} \quad |0\rangle \quad & \quad\quad |1\rangle \quad \end{array} & \end{array}$$

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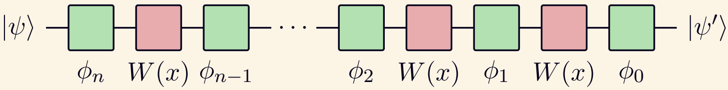

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Later: the theory of *block encodings*!

⚛️ The BBl Protocol[1] as QSP

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⚛️ The BB1 Protocol[1] as QSP

$$U(\theta) = e^{i\pi\sigma_z/2} V(\theta) e^{-i\eta\sigma_z} V(\theta) e^{2i\eta\sigma_z} V^2(\theta) e^{-2i\eta\sigma_z} V(\theta) e^{i\eta\sigma_z}$$

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For $\eta = (1/2)\cos^{-1}(-1/4) \approx 0.912$ this yields:

$$U(\cos^{-1} x) = \begin{bmatrix} P(x) & i\sqrt{1-x^2}Q(x) \\ * & * \end{bmatrix},$$

$$|P(x)|^2 = \frac{1}{8}(30x^2 - 45x^4 + 35x^6 - 15x^8 + 3x^{10}).$$

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For small θ, $|\langle 0|U(\theta)|0\rangle|^2$ expands $1 + (5/512)\theta^6 + \ldots$

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Achievable polynomials

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Two components can be chosen (mostly) freely; e.g., $C(x) = 0$, and $|A(x)|^2 \leq 1$ on $x \in [-1, 1]$; measurement in $|\pm\rangle$ basis yields $p_+ = |A(x)|^2$

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⚠️ For other parameterizations ($x \mapsto [z + z^{-1}]/2$) or ansätze, this can change!

QSP ansätze and functional constraints

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The circuit depth for P of degree d is d. Moreover, $x \mapsto -x$ changes signs of P, Q, and $x \in \{\pm 1, 0\}$ induce constraints

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Finding A, C reduces to *norm-constrained polynomial approximation or interpolation!*

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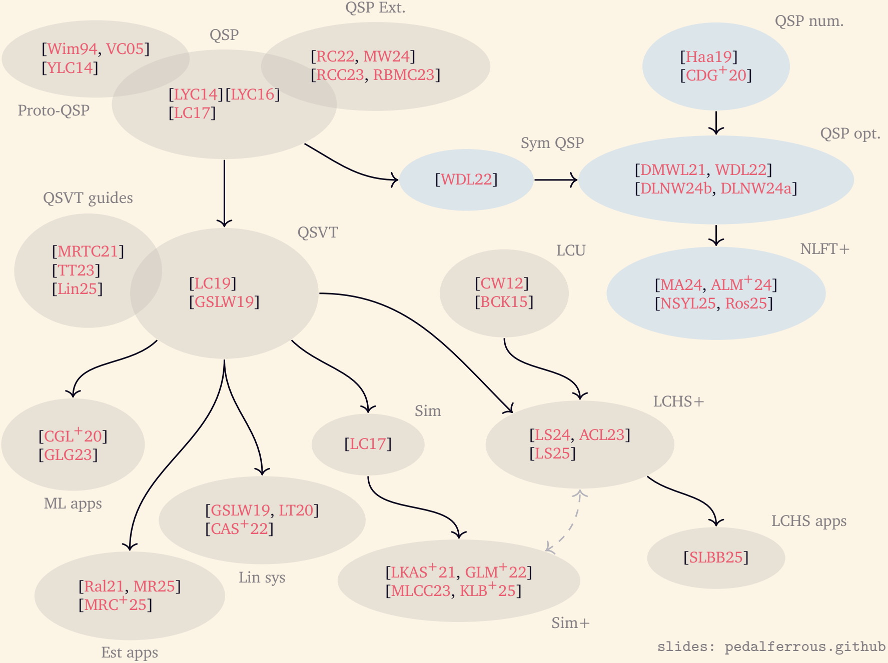

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✨ pyQSP: `https://github.com/ichuang/pyqsp`.

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What we want to know: How can QSP *usefully* transform quantum systems comprising *many qubits*?

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Up next: finding qubits inside large unitaries

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$$-S_s(\pi)S_t(\pi) = \begin{bmatrix} -1+2\xi^2 & -2\xi\sqrt{1-\xi^2} \\ -2\xi\sqrt{1-\xi^2} & 1-2\xi^2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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$$U_s(\alpha) = e^{i\alpha(2|s\rangle\langle s| - I)}$$

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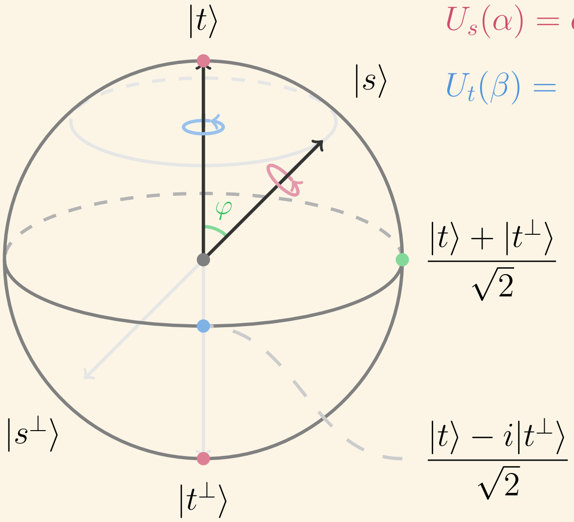

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Let $B_{L,1} \in \mathbb{C}^{d\times r}$, $B_{R,1} \in \mathbb{C}^{d\times c}$ have *orthonormal columns*.

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A unitary $U \in \mathbb{C}^{d\times d}$ is an (α, ε)-block encoding of A if

$$\left\| A - \alpha B_{L,1}^{\dagger} U B_{R,1} \right\|_{\mathrm{op}} \leq \varepsilon.$$

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E.g., a $(1,0)$ block encoding, with $B_L = (B_{L,1}, B_{L,2})$ and $B_R = (B_{R,1}, B_{R,2})$ the unitary completions of $B_{L,1}, B_{R,1}$:

$$B_L^{\dagger} U B_R = \begin{bmatrix} A & * \\ * & * \end{bmatrix}, \quad B_L^{\dagger}(\Pi_L U \Pi_R) B_R = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

An (α, a, ε)-block encoding of A [GSLW19]

$$\left\| A - \alpha(\langle 0|^{\otimes a} \otimes I)U(|0\rangle^{\otimes a} \otimes I) \right\| \le \varepsilon,$$

where A is s-qubit, and U is $(s+a)$ qubit, and, e.g., $(\langle 0|^{\otimes a} \otimes I) = B_{L,1}^{\dagger}$ and $\Pi_L = |0\rangle\langle 0|^{\otimes a} \otimes I$.

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Observe action of U on a **right** singular vector

$$U|\psi_i\rangle = \Pi_L U \Pi_R |\psi_i\rangle + (1 - \Pi_L) U |\psi_i\rangle \tag{1}$$

$$= \xi_i |\psi_i\rangle + \sqrt{1 - \xi_i^2}\, |*\rangle \tag{2}$$

Let Π_L, Π_R, U, A as before, ξ_i the i-th singular value of A, k the largest index where $\xi_k = 1$, and $r = \mathrm{rank}(A)$.

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$$\begin{aligned}
&\mathscr{H}_i = \mathrm{span}(|\psi_i\rangle), && \tilde{\mathscr{H}}_i = \mathrm{span}(|\tilde{\psi}_i\rangle), && i \in [k], \\
&\mathscr{H}_i = \mathrm{span}(|\psi_i\rangle, |\psi_i^\perp\rangle), && \tilde{\mathscr{H}}_i = \mathrm{span}(|\tilde{\psi}_i\rangle, |\tilde{\psi}_i^\perp\rangle), && i \in [r]\backslash[k], \\
&\mathscr{H}_i^R = \mathrm{span}(|\psi_i\rangle), && \tilde{\mathscr{H}}_i^R = \mathrm{span}(U|\psi_i\rangle), && i \in [d]\backslash[r], \\
&\mathscr{H}_i^L = \mathrm{span}(U^\dagger|\tilde{\psi}_i\rangle), && \tilde{\mathscr{H}}_i^L = \mathrm{span}(|\tilde{\psi}_i\rangle), && i \in [\tilde{d}]\backslash[r].
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Qubitization: we have to show $\mathscr{H}_i$, $\mathscr{H}_i^R$, and $\mathscr{H}_i^L$ and so on are pairwise orthogonal:

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$$\langle \psi_i^\perp | \psi_j^\perp \rangle = \langle \tilde{\psi}_i^\perp | \tilde{\psi}_j^\perp \rangle = \delta_{ij}, \quad (\mathscr{H}_i)^\perp \perp (\tilde{\mathscr{H}}_j)^\perp, \tag{5}$$

$$\langle \psi_i | \psi_j^\perp \rangle = \langle \tilde{\psi}_i | \tilde{\psi}_j^\perp \rangle = 0, \quad */* \perp */*, \tag{6}$$

$$\langle \psi_i | U^\dagger | \tilde{\psi}_j \rangle = 0, \quad U^\dagger | \tilde{\psi}_j \rangle \in \mathscr{H}_j^L, \tag{7}$$

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The first three from singular vector orthogonality; note that $\langle \tilde{\psi}_i | U \Pi_R U^\dagger | \tilde{\psi}_j \rangle$ can be replaced by $\langle \tilde{\psi}_i | A A^\dagger | \tilde{\psi}_j \rangle$. The final three follow from definition of projectors.

The components of the QSVT circuit[1]

$$U = \bigoplus_{i\in[k]} [\xi_i]_{\tilde{\mathscr{H}}_i}^{\mathscr{H}_i} \oplus \bigoplus_{i\in[r]\setminus[k]} \begin{bmatrix} \xi_i & \sqrt{1-\xi_i^2} \\ \sqrt{1-\xi_i^2} & \xi_i \end{bmatrix}_{\tilde{\mathscr{H}}_i}^{\mathscr{H}_i} \oplus$$

$$\bigoplus_{i\in[d]\setminus[r]} [1]_{\tilde{\mathscr{H}}_i^R}^{\mathscr{H}_i^R} \oplus \bigoplus_{i\in[\tilde{d}]\setminus[r]} [1]_{\tilde{\mathscr{H}}_i^L}^{\mathscr{H}_i^L} \oplus [*]_{\tilde{\mathscr{H}}^\perp}^{\mathscr{H}^\perp},$$

$$e^{i\phi(2\Pi_R - I)} = \bigoplus_{i\in[k]} [e^{i\phi}]_{\mathscr{H}_i}^{\mathscr{H}_i} \oplus \bigoplus_{i\in[r]\setminus[k]} \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}_{\mathscr{H}_i}^{\mathscr{H}_i} \oplus$$

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[1]Our notation maps top subspace to bottom: $\mathbf{r} \downarrow \mathbf{g}$

***QSVT* [GSLW19]:** Let $\Phi = \{\phi_j\}_{j\in[n]} \in \mathbb{R}^n$; the QSVT protocol associated with Φ and block encoding U has circuit (taking n odd):

$$U_\Phi \equiv e^{i\phi_1(2\Pi_L - I)} U \prod_{j=1}^{(n-1)/2} e^{i\phi_{2j}(2\Pi_R - I)} U^\dagger e^{i\phi_{2j+1}(2\Pi_L - I)} U.$$

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$$U_\Phi = \bigoplus_{i\in[k]} [P(\xi_i)]_{\tilde{\mathscr{H}}_i}^{\mathscr{H}_i} \oplus \bigoplus_{i\in[r]\setminus[k]} \begin{bmatrix} P(\xi_i) & * \\ * & * \end{bmatrix}_{\tilde{\mathscr{H}}_i}^{\mathscr{H}_i} \oplus \bigoplus_{i\in[d]\setminus[r]} [e^{i\phi_0}]_{\tilde{\mathscr{H}}_i^R}^{\mathscr{H}_i^R} \oplus \bigoplus_{i\in[\tilde{d}]\setminus[r]} [e^{-i\phi_0}]_{\tilde{\mathscr{H}}_i^L}^{\mathscr{H}_i^L} \oplus [*]_{\tilde{\mathscr{H}}^\perp}^{\mathscr{H}^\perp},$$

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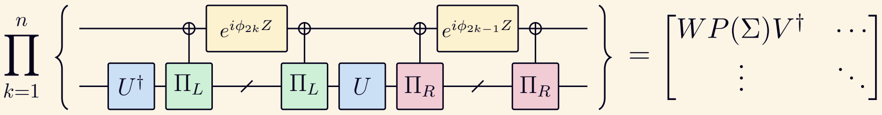

Another approach:[1] *the cosine-sine decomposition*

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It turns out one can produce *simultaneous* SVDs for multiple unitaries at once: $U_{ij} = V_i D_{ij} W_j^\dagger$ [TT23, PW94]

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Let $U \in \mathbb{C}^{d\times d}$ a unitary matrix partitioned into blocks of size $\{r_1, r_2\} \times \{c_1, c_2\}$:

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \text{ where } U_{ij} \in \mathbb{C}^{r_i \times c_j},$$

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Then there exist unitaries $V_i \in \mathbb{C}^{r_i \times r_i}$ and $W_j \in \mathbb{C}^{c_j \times c_j}$ s.t.

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} W_1 & \\ & W_2 \end{bmatrix}^\dagger,$$

where each D_{ij} is diagonal in $\mathbb{C}^{r_i \times c_j}$, possibly zero-padded.

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Specifically, we can write D as:

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & & & I & & \\ & C & & & S & \\ & & I & & & 0 \\ \hline I & & & 0 & & \\ & S & & & -C & \\ & & 0 & & & -I \end{array}\right],$$

$$= \underbrace{\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}}_{\mathscr{X}_0 \to \mathscr{Y}_0} \oplus \underbrace{\begin{bmatrix} C & S \\ S & -C \end{bmatrix}}_{\mathscr{X}_C \to \mathscr{Y}_C} \oplus \underbrace{\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}}_{\mathscr{X}_1 \to \mathscr{Y}_1}.$$

where C, S are square w/ entries in $(0,1)$, and $C^2 + S^2 = I$

Cosine-sine decomposition proof sketch

1. Start with the SVD of one block $U_{11} = V_1 D_{11} W_1^\dagger$

2. Compute QR decompositions of $U_{21} W_1$ and $U_{12}^\dagger V_1$, giving V_2, W_2 to make these operators upper-diagonal with nonnegative diagonal entries:

$$\begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}^\dagger \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} W_1 & \\ & W_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & V_2^\dagger U_{22} W_2 \end{bmatrix}.$$

3. Observing the rest of the unitary (whose rows and columns must be orthonormal), this forces the entries of D_{12}, D_{21} to be diagonal and $C^2 + S^2 = I$

4. Choose $W_2 \mapsto W_2'$ to correct D_{22} (free up to unitary).

The limits of qubitization

⏱️

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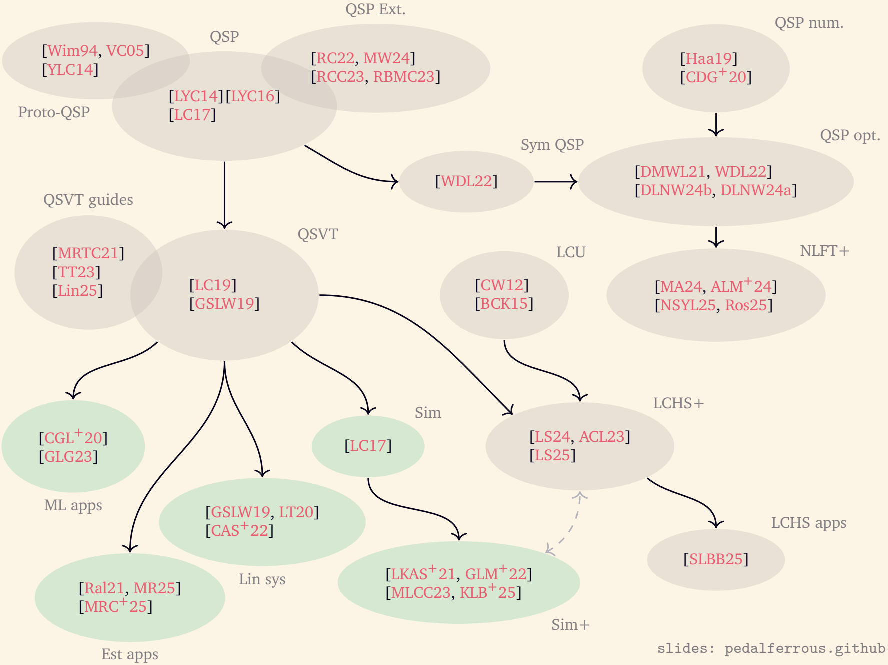
Question: Weaker qubitization? Can we transform Jordan blocks [LS24] ? Irrep labels? Oracle-marked subspaces?


Proto-QSP
[Wim94, VC05]
[YLC14]
QSP
[LYC14][LYC16]
[LC17]
QSP Ext.
[RC22, MW24]
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QSP num.
[Haa19]
[CDG+20]
Sym QSP
[WDL22]
QSP opt.
[DMWL21, WDL22]
[DLNW24b, DLNW24a]
QSVT guides
[MRTC21]
[TT23]
[Lin25]
QSVT
[LC19]
[GSLW19]
LCU
[CW12]
[BCK15]
NLFT+
[MA24, ALM+24]
[NSYL25, Ros25]
ML apps
[CGL+20]
[GLG23]
Sim
[LC17]
LCHS+
[LS24, ACL23]
[LS25]
Lin sys
[GSLW19, LT20]
[CAS+22]
Est apps
[Ral21, MR25]
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Matrix functions for large* linear operators

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Low energy projection:
Input **Hamiltonian**, **apply bandpass function**
Inversion:
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🌞 ***Changing the polynomial changes the algorithm*** 🌞

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QSVT in practice

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Simulation

Linear system solving

Estimation tasks

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⚠️ QSVT captures BQP-complete problems; if you ask for too much, it'll be expensive

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⚠️: Constant factors seem to matter [KREO25]; error propagation requires care [TT23, GSLW19]

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Beyond instantiation, diverse methods for (approximately) combining block encodings [LW19, VG25]

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👉 **A clean abstraction**; use of diverse functional analytic, representation theoretic, and numerical linear algebraic tools is made clear

📌 A note on dequantization and QSVT

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Moral: If matrix is low rank, block encoding is QRAM-based,[1] or your required precision is constant, then sketching methods work [CGL+20, GLG23, EHG25]

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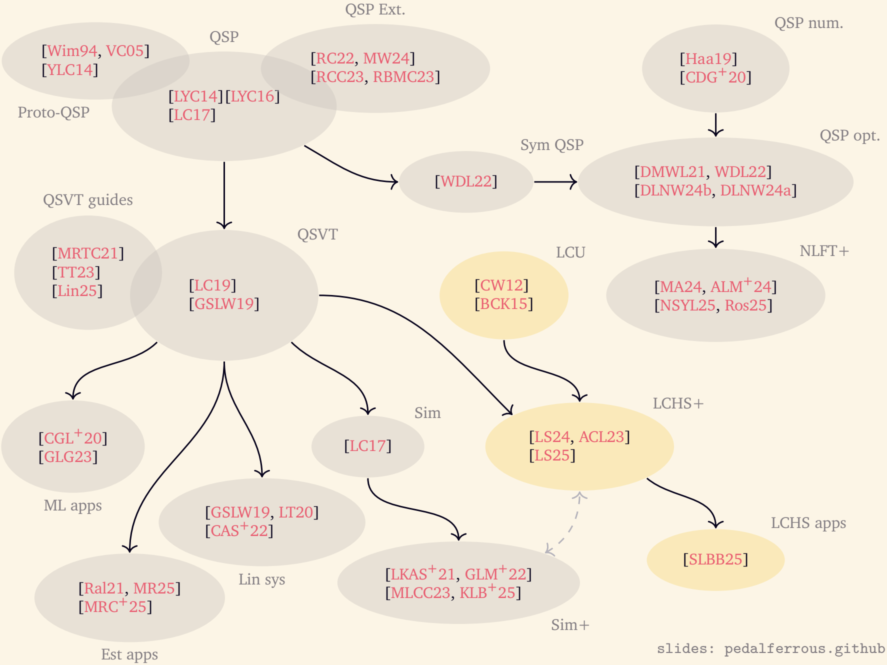
(Over)sampling and query access: analogues to quantum state preparation assumptions

In practice, there may be large polynomial separations even in these cases, but improvement is rapid!

[1] E.g., recommendation systems, principal component analysis, supervised clustering, support vector machines, low-rank regression, semidefinite program solving, etc.



Proto-QSP
[Wim94, VC05]
[YLC14]
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[LYC14][LYC16]
[LC17]
QSP Ext.
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QSVT guides
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QSVT
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[GSLW19]
LCU
[CW12]
[BCK15]
NLFT+
[MA24, ALM+24]
[NSYL25, Ros25]
ML apps
[CGL+20]
[GLG23]
Sim
[LC17]
LCHS+
[LS24, ACL23]
[LS25]
Lin sys
[GSLW19, LT20]
[CAS+22]
Est apps
[Ral21, MR25]
[MRC+25]
Sim+
[LKAS+21, GLM+22]
[MLCC23, KLB+25]
LCHS apps
[SLBB25]

👪 The great family of block encoding algorithms

QSP [LC17], QSVT [GSLW19], QEP [LS24], G-QSP [MW24], M-QSP [RC22], LCU [CW12], LCHS [ACL23], randomized, parallelized, and hybrid variants!

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Unifying aspect: precise control of subsystem dynamics

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Unifying aspect: precise control of subsystem dynamics

⚠️ Comparison not always possible, and choices between these can depend on architecture and instance size!

Linear combination of unitaries (LCU) [CW12, BCK15]

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Let $V = \sum_k \alpha_k U_k$ for $\alpha_k > 0$ and U_k unitary. Let W be any unitary that acts as

$$W|0\rangle = \frac{1}{\sqrt{\alpha}} \sum_k \sqrt{\alpha_k}\,|k\rangle, \quad \alpha \equiv \sum_k \alpha_k.$$

Then

$$W^\dagger \left[\sum_k |k\rangle\langle k| \otimes U_k \right] W \equiv W^\dagger U W$$

is an $(\alpha, *, 0)$ block encoding of V. Often W, U are referred to as the PREPARE and SELECT operations.

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Desired transformation written as a sum of block-encoded unitaries $\implies$ **LCU describes costs**

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Time-independent Hamiltonian sim for d-term LCU

Method	aux qubits	query comp
LCU	$\lceil \log d \rceil \log(\alpha t/\varepsilon)$	$\lceil \alpha t \rceil \log(\alpha t/\varepsilon)$
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QSVT insight: quick approximation of entire functions

$$\cos(xt) = J_0(t) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(t) T_{2k}(x),$$

$$\sin(xt) = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(t) T_{2k+1}(x),$$

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Approximating $u(t)$ relies on time-ordered integral:

$$u(t) = U_0(t)u_0 + \int_0^t U_s(t)b(s)\,ds,$$
$$U_s(t) \equiv \mathcal{T}\exp\left[\int_s^t A(s')\,ds'\right].$$

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$$\forall x \geq 0,\ \ e^{-x} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(k)\,e^{-ikx}\,dk$$

$$\Longrightarrow \ \forall t \geq 0,\ \ e^{-(L+iH)t} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(k)\,e^{-i(kL+H)t}\,dk.$$

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⚠️ **Integral quadrature can be technical [ACL23, TW14]**

Beyond e^{-x}, approximations for $1/x$ permit linear matrix equation solvers [SLBB25], e.g., $AX - XB = C$

The quantum linear system problem: $Ax = b$

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Algorithm	*Query comp.*	*N.b.*
HHL [HHL09]	$O(\kappa^2/\varepsilon)$	VTAA
LCU [CKS17]	$O(\kappa^2 \operatorname{polylog}(\varepsilon^{-1}))$	VTAA
QSVT [GSLW19]	$O(\kappa^2 \log(\varepsilon^{-1}))$	
VTAA [Amb10]	$O(\kappa \operatorname{polylog}(\kappa\varepsilon^{-1}))$	Overhead
Adiabatic [CAS+22]	$O(\kappa \operatorname{polylog}(\varepsilon^{-1}))$	
Eig. filter [LT20]	$O(\kappa \log(\varepsilon^{-1}))$	Adiabatic

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Moral: block encoding A^{-1} requires subnormalizing by κ, meaning $O(\kappa)$ amp, and thus $O(\kappa^2)$ query comp.

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⚠️ **The utility of $\log(1/\varepsilon)$ error dependence outside subroutines is debatable**

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🚡 Open avenues in the near-term

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Moving beyond spectral mapping	Weaker access models and hybrid methods	More flexible proof techniques

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Past examples include **LCHS** [LS25], **multiproduct formulas** [LKW19], and **SKTs for QSP** [Ros25]

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Moving beyond spectral mapping	Weaker access models and hybrid methods	More flexible proof techniques

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We need more *end-to-end analysis* for *realistic problem instances*! Most operations can be fuzzed!

🤖 Block encoding algorithms & machine learning

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Great platform for dequantization efforts [CGL+20]

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Plenty of work left in realizing matrix manipulations efficiently (e.g., inference step in transformers [GYC+25])

🤖 Block encoding algorithms & machine learning

Great platform for dequantization efforts [CGL+20]

Investigations into quantum analogues of universal approximators [[illegible], GC+21, PSCLGFL20, Ros25]

⚠️ **As always, *caveat lector***

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🏡 Take-home

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We have *limited tools beyond qubitization* for nonlinear transformations of subsystem dynamics

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These algorithms and their hybrids enable importation of *vast techniques from applied mathematics*

We're not yet sufficiently optimistic about these algorithms!

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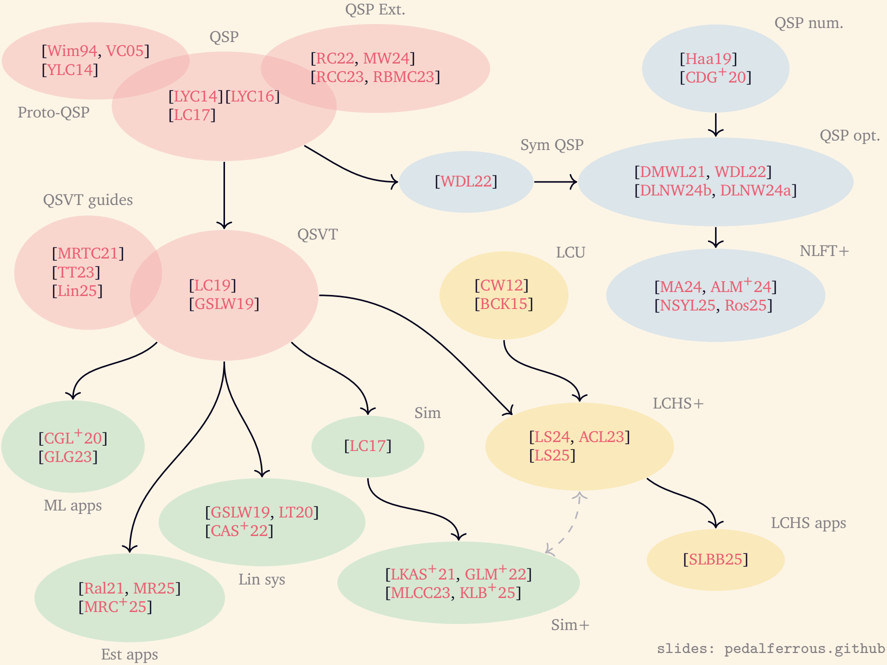
✨ slides++



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[CW12]
[BCK15]
NLFT+
[MA24, ALM+24]
[NSYL25, Ros25]
[CGL+20]
[GLG23]
ML apps
Sim
[LC17]
LCHS+
[LS24, ACL23]
[LS25]
[GSLW19, LT20]
[CAS+22]
Lin sys
[LKAS+21, GLM+22]
[MLCC23, KLB+25]
Sim+
LCHS apps
[SLBB25]
[Ral21, MR25]
[MRC+25]
Est apps

Szegő functions

A real-valued measureable even function $f : [0,1] \to [-1,1]$ is called Szegő if it satisfies

$$\int_0^1 \log\left|1 - f(x)^2\right| \frac{dx}{\sqrt{1-x^2}} > -\infty \tag{9}$$

Szegő functions come with a norm showing they're a subset of square-summable

The work of [ALM+24] shows that Szegő functions satisfying $\|f\|_\infty \leq 1-\eta$ admit unique QSP phases, and produce a unitary with a matrix element converging to f in a nice way, and that

$$\|\Phi - \Phi'\|_\infty \leq O(\eta^{-3}) \|f - f'\|_S. \tag{10}$$

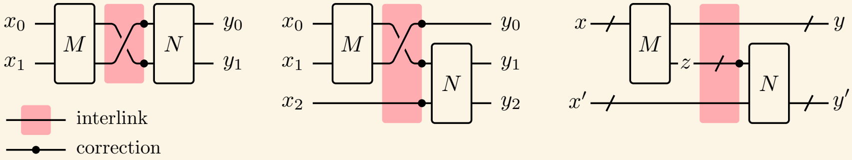
👹 Hybrid methods in block encoding algorithms

Hybrid oscillator-qubit systems in experiment [LSS+25]

LCU of product formulas [CST+21, LKW19]

Parallel, randomized, and self-composed QSP/QSVT [RCC23, MRC+25, MR25]

Approximate or accelerated products for block encodings [LW19, VG25]



On the optimality of QSP/QSVT

Lower bound for eig. transformation; Thm. 73 [GSLW19]
Let $I \subseteq [-1,1]$, $a \geq 1$ and suppose U is a $(1,a,0)$-block encoding of an unknown Hermitian matrix H with the promise that the spectrum of H lies within I. Let $f: I \to \mathbb{R}$, and suppose access to a quantum circuit V that implements a $(1,b,\varepsilon)$-block encoding of $f(H)$ using T applications of U for all U satisfying the promise. Then for all $x \neq y \in I \cap [-1/2, 1/2]$ we have that

$$T = \Omega\left[\frac{|f(x) - f(y)| - 2\varepsilon}{|x - y|}\right]$$

Lower bound for quantum matrix functions; [MS24]
For any continuous function $f(x): [-1,1] \to [-1,1]$, there is a 2-sparse Hermitian matrix A with $|A| \leq 1$ and two indices i,j such that $\Omega(\widetilde{\deg}_\varepsilon(f))$ queries to A are required in order to compute $\langle i|f(A)|j\rangle \pm \varepsilon/4$.