Quantum element-wise transforms

Zane Rossi[1], Rahul Sarkar[2]

The University of Tokyo[1]
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or, QEWTs 

[2606.06456]

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The result — a growing, numerically-stable toolkit to interpret, optimize, and communicate quantum algorithms

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$$e^{(A+iB)t} = \int f(k) e^{(kA+B)t}\, dk$$

$$(A, B) \mapsto f(A, B)$$

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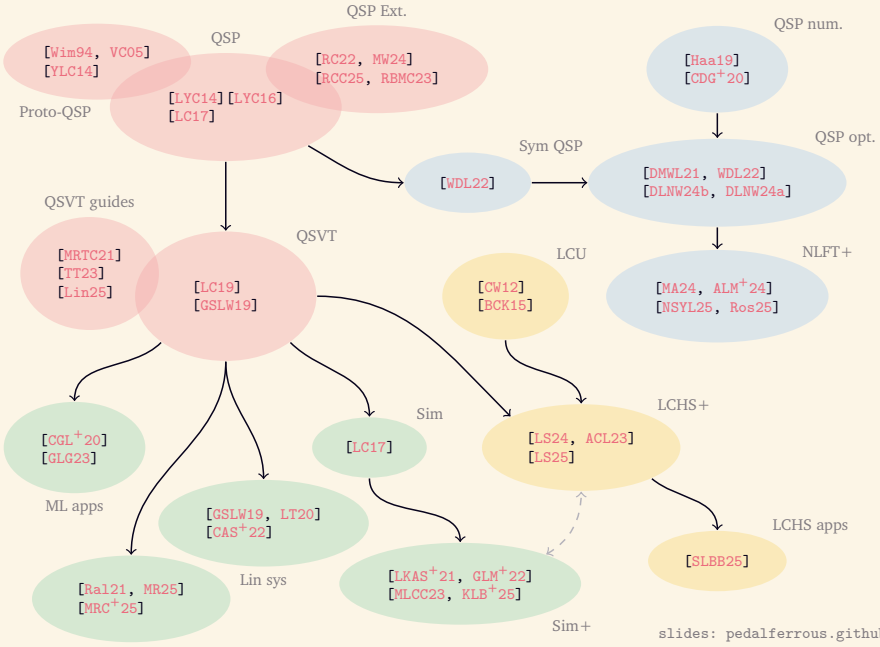
These methods are incredibly flexible, and allow importation of/comparison with diverse classical techniques!



slides: pedalferrous.github.io

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$$\left\| (\langle 0^a | \otimes I_n) U (|0^a\rangle \otimes I_n) - A/\alpha \right\| \leq \varepsilon$$

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Algorithms like QSVT[1] [GSLW19] and LCU[2] [CW12] allow spectral maps and simple matrix arithmetic; e.g., $A \mapsto f(A)$, or $(A, B) \mapsto \gamma A + \delta B$

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[2] Linear combination of unitaries

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For BE algorithms, these relate to concrete questions in functional analysis and numerical linear algebra!

A note on matrix algebras

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This work concerns the efficient realization of various matrix manipulations by quantum algorithms

A note on matrix algebras

Square matrices over $\mathbb{C}$ admit associative algebra, i.e., addition, multiplication, scalar multiplication

$$C_{ij} = \sum_k A_{ik} B_{kj} \leftrightarrow C = AB$$

A note on matrix algebras

Functions of matrices are often defined spectrally, and algorithms like QSVT [GSLW19, MRTC21] act on the singular values, e.g., if $M = U\Sigma V^\dagger$

$$f^{(\mathrm{SV})}(M) \equiv U f(\Sigma) V^\dagger = U \begin{bmatrix} f(\xi_0) & & \\ & \ddots & \\ & & f(\xi_{n-1}) \end{bmatrix} V^\dagger, \quad \Sigma = \mathrm{diag}(\xi_0, \xi_1, \ldots, \xi_{n-1}),$$

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However, there exist other associative algebras for matrices over fields, including the element-wise product[1]

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This product appears ubiquitously in classical signal processing, machine learning, and statistics, and has no simple spectral interpretation! [HJ85, HJ91, CWP+25, Sty73] E.g., consider $f^{\circ}(A)$

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$$\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0,m-1} \\ a_{10} & a_{11} & & \vdots \\ \vdots & & \ddots & \\ a_{n-1,0} & \cdots & & a_{n-1,m-1} \end{bmatrix} \mapsto \begin{bmatrix} f(a_{00}) & f(a_{01}) & \cdots & f(a_{0,m-1}) \\ f(a_{10}) & f(a_{11}) & & \vdots \\ \vdots & & \ddots & \\ f(a_{n-1,0}) & \cdots & & f(a_{n-1,m-1}) \end{bmatrix}.$$

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Despite this product's ubiquity, quantum algorithms for element-wise manipulations are only cursorily understood [GYC+25] …

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A note on matrix algebras

$\otimes$	`\otimes`	(Kronecker/tensor product),
$\circ$	`\circ`	(Element-wise product),
$\boxtimes$	`\boxtimes`	(Tracy–Singh product),
$\odot$	`\odot`	(Khatri-Rao product),
$*$	`\ast`	(Convolution).

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Give outlook, extensions, and high-level takeaways

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Quantum element-wise transform; *QEWT*[1]

Let f be a polynomial of degree d satisfying $|f| \leq 1$ on $[-1,1]$, with coefficients c_k. Then, given access to controlled U, where U is an (α, a, ε) block encoding of an n-qubit operator A, there exists a quantum circuit preparing a (β, b, δ) block encoding of $f^{\circ}(A/\alpha)$ using d copies of controlled U, where (β, b, δ) satisfy

$$\beta = \sum_{k=1}^{d} |c_k|, \quad b = \mathcal{O}(a + n \log d), \quad \delta = \sum_{k=1}^{d} |c_k| \left[(1+\varepsilon/\alpha)^k - 1 \right].$$

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Main result (Thm. II.3)	$\mathcal{O}(d)$	$\mathcal{O}(a+n(\log d))$
LCU variant (Thm. III.4)	$\mathcal{O}(d)$	$\mathcal{O}(a+n(\log d))$
Approx. var. (Cor. III.5.2)	$\mathcal{O}(d^{\log(\log(d/\varepsilon))})$	$\mathcal{O}(a+n\log(d^{\log(\log(d/\varepsilon))}))$
Half-comp. (Thm. C.1)	$\mathcal{O}(d)$	$\tilde{\mathcal{O}}(a+nd)$

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$$\begin{bmatrix} | & | & | \\ |\psi\rangle & * & * \\ | & | & | \end{bmatrix} \to \begin{bmatrix} \psi_0 & & \\ & \psi_1 & \\ & & \ddots \end{bmatrix} \to \begin{bmatrix} f(\psi_0) & & \\ & f(\psi_1) & \\ & & \ddots \end{bmatrix} \to \begin{bmatrix} | & | & | \\ |f(\psi)\rangle & * & * \\ | & | & | \end{bmatrix}$$

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All of this establishes query lower bounds, and space/success probability upper bounds, but leaves huge room for improvement!

A minimal example

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Note — $A \circ B$ is *principal submatrix*[1] of $A \otimes B$

Prev methods [GYC+25] permute rows/cols of $A^{\otimes d}$ — order nd space!

[1]Common identity, but for introduction see [HJ85, HJ91]

A minimal example

I.e., if we have block encodings of A, B, acting on disjoint registers …

$$U_{A\otimes I} = \left[\begin{array}{cccc|c} a_0 & & a_1 & & \\ & a_0 & & a_1 & * \\ a_2 & & a_3 & & \\ & a_2 & & a_3 & \\ \hline & * & & & * \end{array}\right], \qquad U_{I\otimes B} = \left[\begin{array}{cccc|c} b_0 & b_1 & & & \\ b_2 & b_3 & & & * \\ & & b_0 & b_1 & \\ & & b_2 & b_3 & \\ \hline & * & & & * \end{array}\right],$$

A minimal example

…then a row and column permutation relocates $A \circ B$ to the top-left

$$U_{A\otimes B} = \left[\begin{array}{cccc|c} a_0b_0 & * & * & a_1b_1 & \\ * & * & * & * & \\ * & * & * & * & * \\ a_2b_2 & * & * & a_3b_3 & \\ \hline & * & & & * \end{array}\right] \longrightarrow U_{P(A\otimes B)P^\dagger} = \left[\begin{array}{cccc|c} a_0b_0 & a_1b_1 & * & * & \\ a_2b_2 & a_3b_3 & * & * & \\ * & * & * & * & * \\ * & * & * & * & \\ \hline & * & & & * \end{array}\right].$$

A minimal example

If we then mask unwanted elements by left/right matrix mult …

$$V \equiv \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \Longrightarrow V[P(A \otimes B)P^{\dagger}]V = \begin{bmatrix} a_0 b_0 & a_1 b_1 & & \\ a_2 b_2 & a_3 b_3 & & \\ & & & \end{bmatrix},$$

A minimal example

…and find a way to space-efficiently copy the top-left block, then we can return to input block encoding form $(I \otimes *)$

$$\begin{bmatrix} a_0b_0 & a_1b_1 & & \\ a_2b_2 & a_3b_3 & & \\ & & * & * \\ & & * & * \end{bmatrix} \rightarrow \begin{bmatrix} a_0b_0 & a_1b_1 & & \\ a_2b_2 & a_3b_3 & & \\ & & a_0b_0 & a_1b_1 \\ & & a_2b_2 & a_3b_3 \end{bmatrix} = I \otimes (A \circ B).$$

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caveat lector!

We have to control gate/query complexity, success prob while making sure matrix multiplication and currently unspecified *copy* operation together require only $\log d$ additional space

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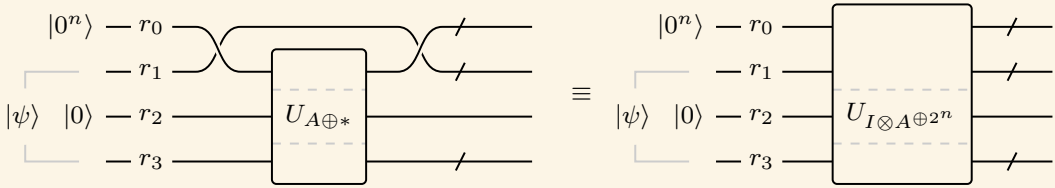

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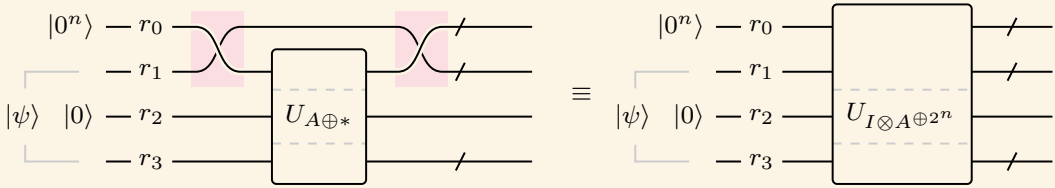

A type of *decontrollization*; i.e., decouple action of block encoding from state in r_1

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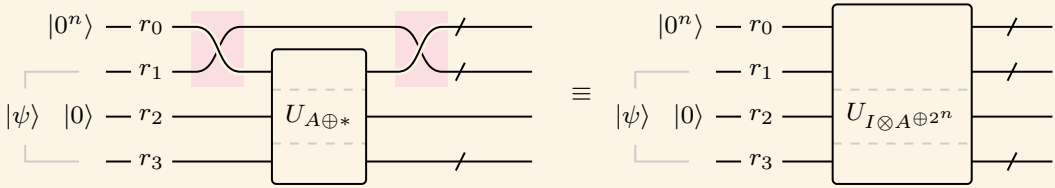

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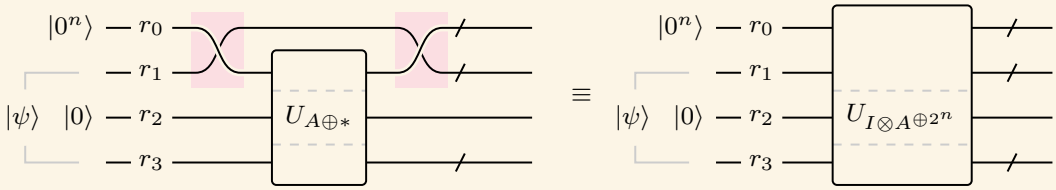

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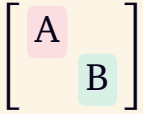
Note 2 — blocks of BEs are *not* a 'resource'; basis dependence means instantiating space instantiates blocks, which can be permuted around!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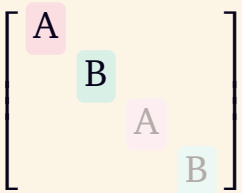
B


A
B
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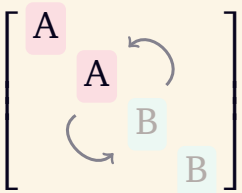
A
A
B
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$$\begin{bmatrix} A & \\ & A \end{bmatrix}$$

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Weaving lemma connects $|r\rangle$-complexity to depth of function call tree

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Nesting swap copy — the weaving lemma

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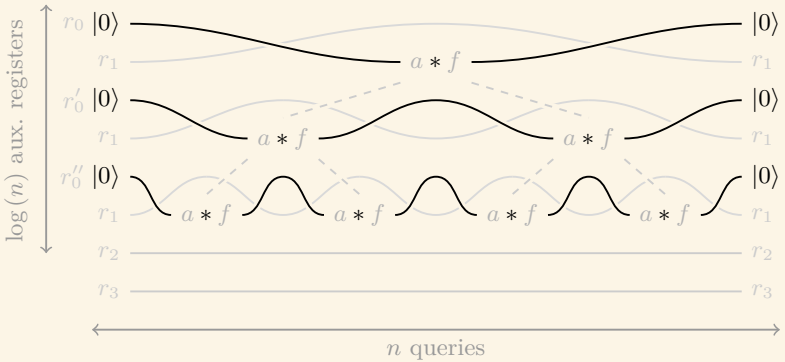

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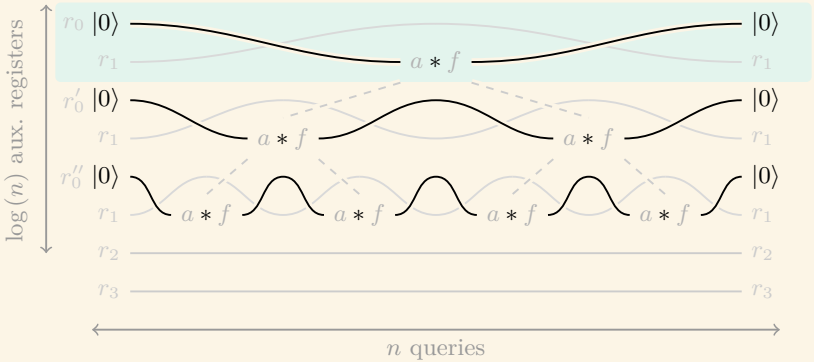

Our *swap copy*-based element-wise product catalytically consumes[1] state in r_0, and can call subroutines which themselves require registers

[1]Here denoted *allocates* and *frees*.

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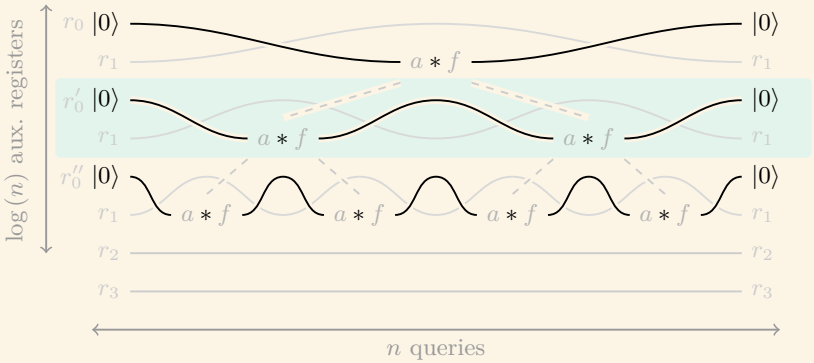

At each level $r_0', r_0'', \ldots$ we will *track* successful BE applications to ensure catalysis — auxiliary space then scales with tree depth!

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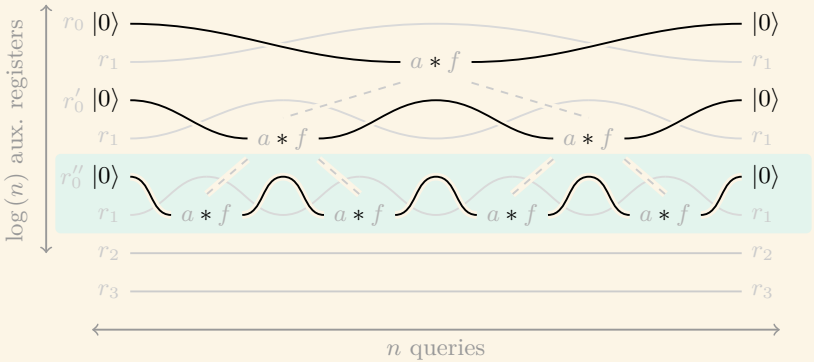

Space-efficient tracking of successful BE application is handled by *compression gadget* techniques [LW19] …

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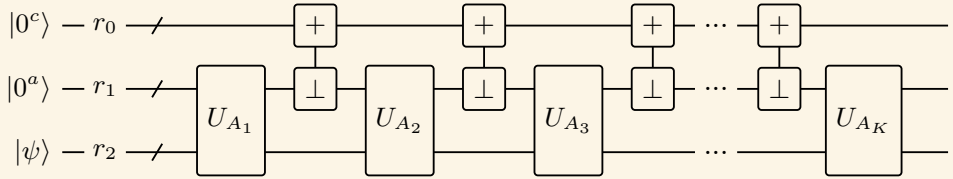

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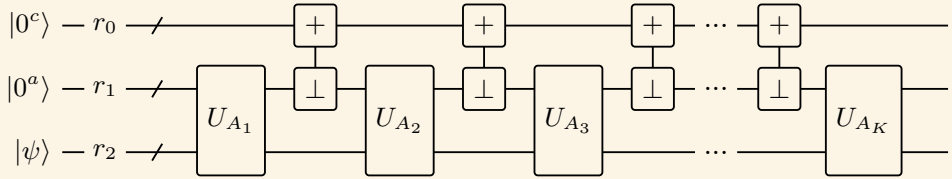

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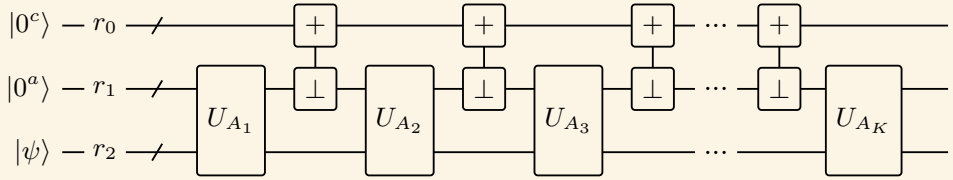

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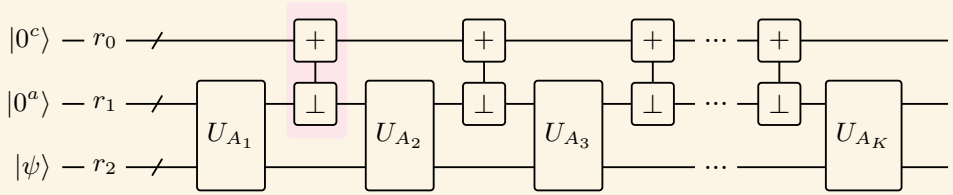

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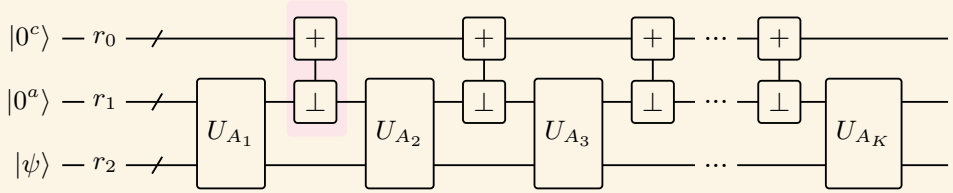

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Linear to logarithmic auxiliary space in product degree!

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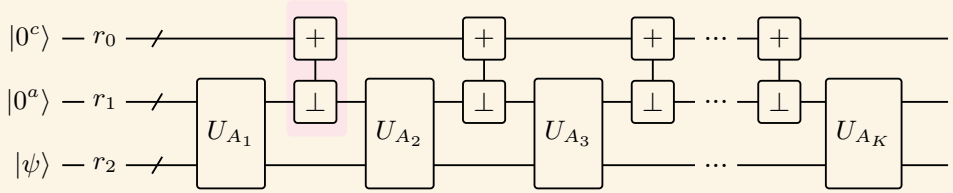

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$$(\langle 0|_{ac} \otimes I_s) V (|0\rangle_{ac} \otimes I_s) = |0\rangle\langle 0|_b \otimes I_s + \sum_{k=0}^{K-1} |k\rangle\langle k|_b \otimes \Big[\prod_{j=0}^{k} A_j\Big]_s,$$

$$W = I_s \oplus \bigoplus_{k=0}^{K-1} \Big[\prod_{j=0}^{k} A_j\Big]_s,$$

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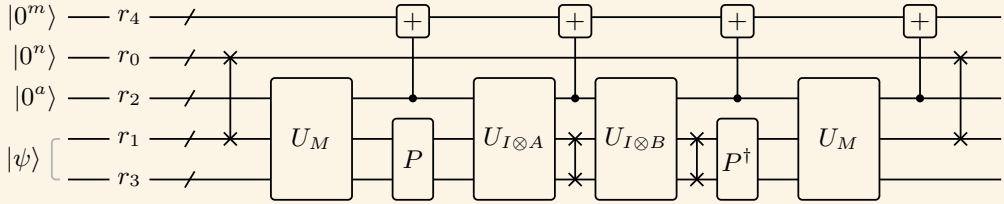

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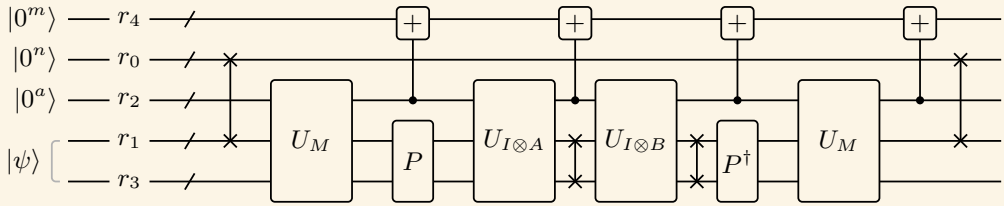

Standard (compressed) method for *matrix product*[1] between M, $P(A \otimes B)P^\dagger$, $M^\dagger$

Followed by *swap copy*, yielding block encoding (BE) of $I \otimes (A \circ B)$

Register r_4 records successful BE application by controlled-additions from r_2. Here r_1 and r_3 store product, and $r_0 \leftrightarrow r_1$ is *swap copy*

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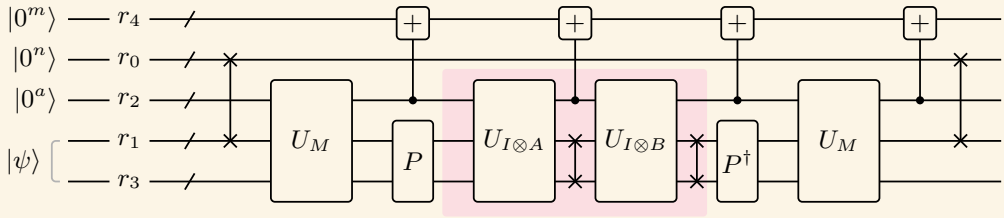

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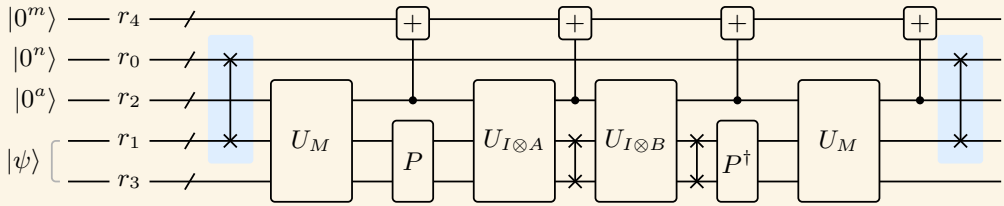

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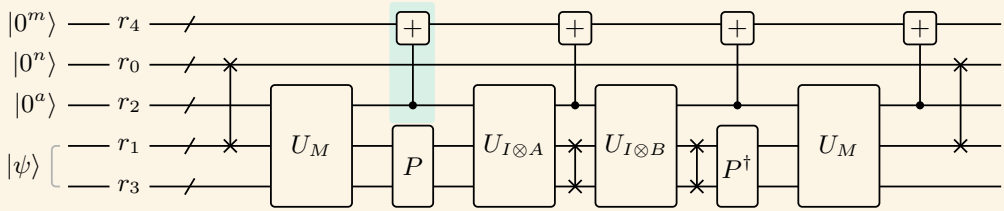

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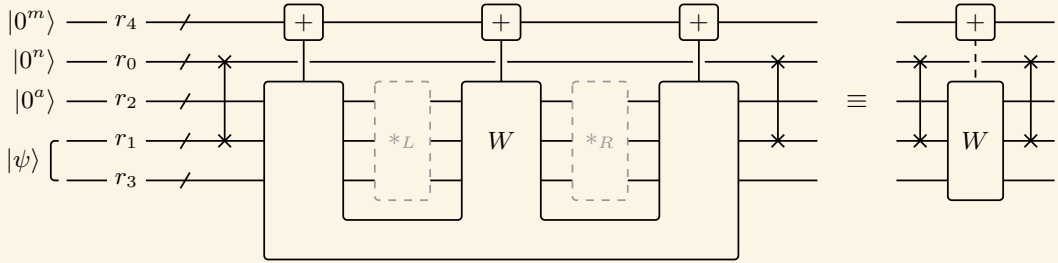

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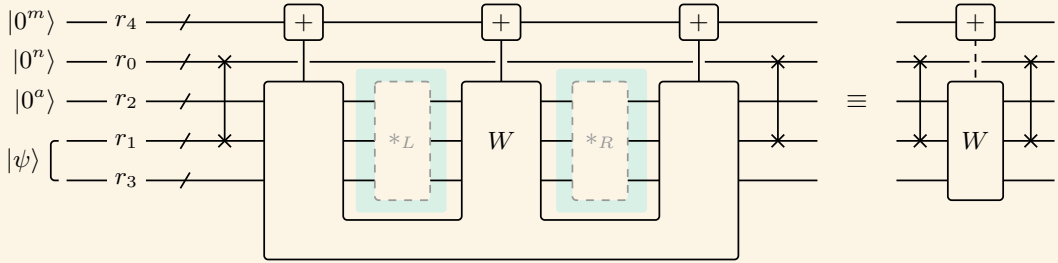

We can recast the base case as a two-slot function taking block encodings to block encodings with form $(I \otimes *)$

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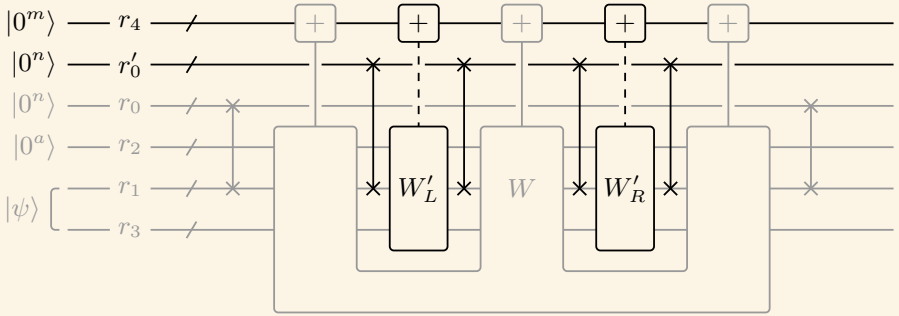

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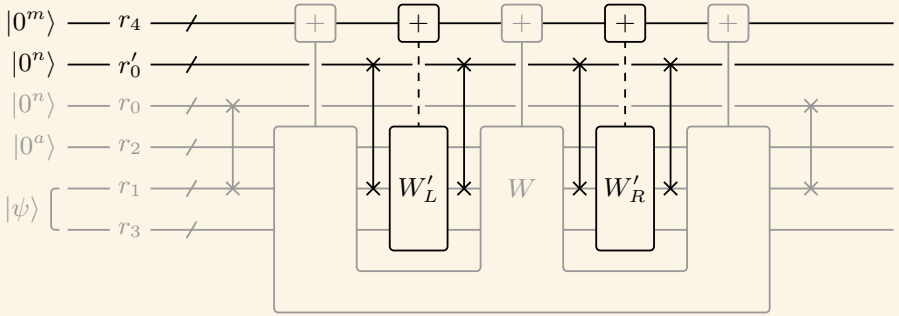

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We recurse on both $*_R$, $*_L$ for $A^{\circ d}$ in $\log d$ depth (and thus aux space!)

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This needs controlled BEs, and linear query complexity in maximum degree

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$$V = \sum_{k=1}^{d} |k\rangle\langle k|_{r_5} \otimes \left[U_{A^{\circ k}}\right]_{r_0 r_1 r_2 r_3 r_4} + \left[I_{r_5} - \sum_{k=1}^{d} |k\rangle\langle k|_{r_5}\right] \otimes I_{r_0 r_1 r_2 r_3 r_4}.$$

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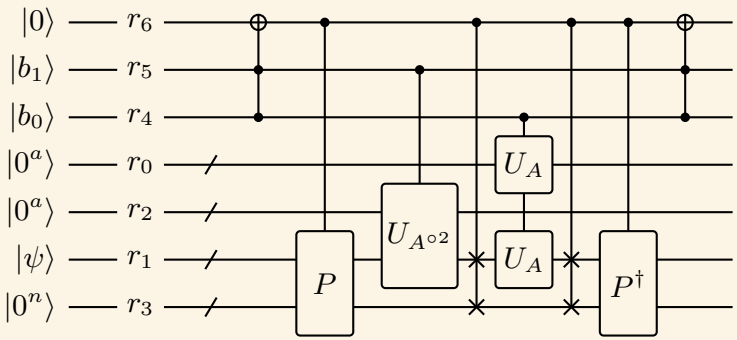

Note — the permutations and swaps are now controlled on the `XNOR` of the control bits; this reduces back to matrix product

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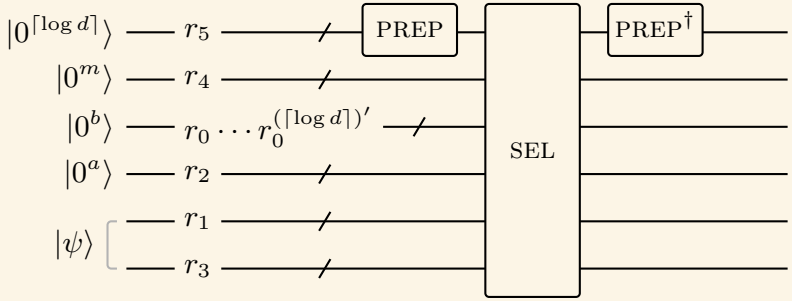

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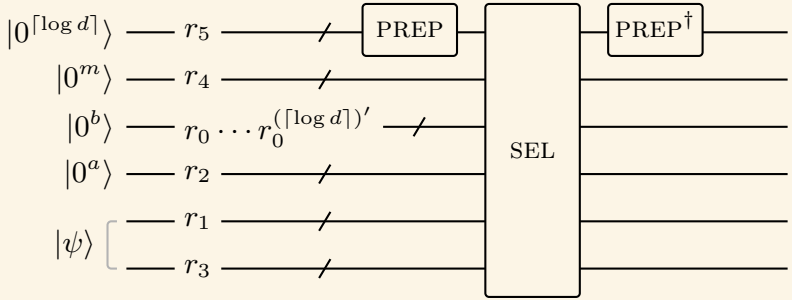

Degree d LCU requires $\log d$ space, and introduces subnormalization, error relating to ℓ_1 norm of coefficients …

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While element-wise product has unique properties, it is a submatrix of a matrix product, meaning subnorm and error propagation are bounded above by simple arguments

Amplified QEWTs

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Let U_{A_k} be $(\alpha_k, a_k, \varepsilon_k \|A_k\|)$ block encodings of (possibly non-unitary) A_k for $k \in \{0, 1, \ldots, K-1\}$ and $\sum_k \varepsilon_k \leq 1/2$. Then for any $0 < \varepsilon < 1/2K$ we can construct an $(\alpha', a', \varepsilon')$ block encoding of the matrix product $A_{K-1} \cdots A_1 A_0$, where α' satisfies

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Moreover, this amplification procedure uses $\mathcal{O}(\alpha_k/(\delta\|A_k\|) \log(\alpha_k/(\varepsilon\|A_k\|)))$ copies of each controlled-U_k and its inverse.

Applications

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For common precisions, this is easily 15–20× space saving!

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$$(h * g)_{\text{circ}} = F^{\dagger}([FhF^T] \circ [FgF^T])F^* = F^{\dagger}([FhF] \circ [FgF])F^{\dagger},$$

2D convolutions

Element-wise products in Fourier basis correspond to convolution in standard basis — this holds for tensors! [HJ85]

$$(h * g)_{\mathrm{circ}}(k_1, k_2) \equiv \sum_{j_1=0}^{K-1} \sum_{j_2=0}^{K-1} h(j_1, j_2) g(k_1 - j_1 \pmod K, k_2 - j_2 \pmod K),$$

$$(h * g)_{\mathrm{circ}}(k_1, k_2) = \mathscr{F}^{-1}(H \circ G) = \mathscr{F}^{-1}(\mathscr{F}(h) \circ \mathscr{F}(g)),$$

$$(h * g)_{\mathrm{circ}} = F^{\dagger}([FhF^T] \circ [FgF^T])F^* = F^{\dagger}([FhF] \circ [FgF])F^{\dagger},$$

This gives a depth $n^2 \log n$ algorithm for computing circular convolution: (α, a, ε), $(\beta, b, \delta) \mapsto (\alpha\beta, \max(a, b) + 1, \alpha\delta + \beta\varepsilon + \varepsilon\delta)$

Non-standard matrix products

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We provide simple quantum algorithms for these products/their hybrids with LCU and QSVT, all of them space-efficient

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Incorporating randomization [MR25] parallelization [MRC+25] to further optimize to specific hardware

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Fundamentally distinct from QSVT or LCU, relying on novel subroutines for space-catalytically transforming block encodings with block structure

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Cheap nonlinear transform for high-dim data, ubiquitous to machine learning, signal processing, and statistics

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🏠 Take-home

We establish a firm foundation for the theory of quantum element-wise transforms (QEWTs), rectifying errors in prior work, and reducing space use exponentially in the degree of the applied function[1]

Quantum algorithms for numerical linear algebra are only cursorily understood, and rapidly developing!

[1] Comparable in cost to other BE algorithms!

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Bonus slides